on the phase space in FPU lattices

Ken'ichi Fukamachi *

Department of Physics, Tokyo Institute of Technology

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Abstract

We investigate $f^{-\mu}$ fluctuations in the Fermi-Pasta-Ulam equation ,which is one of the simplest lattice models, to describe fundamental properties of lattice vibrations. Based on Lyapunov Spectrum, we classify the motion in the phase space and have found an onion-like structure.

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*The conventional mailing address: Oh-okayama, Meguro-ku,Tokyo 152, Japan, FAX:81-3-5734-2739, Email: fukachan@phys.titech.ac.jp and URL: http://www.phys.titech.ac.jp/uja/.

$f^{-\mu}$ Power Spectrum Density(PSD) have been found over a wide variety of phenomena, current in several devices, amount of human activities, pattern in DNA and traffic of computer networks [1–8]. Phenomena which show $f^{-\mu}$ fluctuations cover a lot of different scales. $f^{-\mu}$ PSD have been considered as non-stationary phenomena. Especially intermittency behavior [9] is considered important to understand the origin of $f^{-\mu}$ PSD but no established explanation of $f^{-\mu}$ PSD is known so far.

The FPU equation [10] is composed of both harmonic linear terms and small non-linear terms (quadratic, cubic, and broken linear forces). It is simple and general enough to describe the fundamental properties of non-linear lattices.

In the previous letter [1] we have clarified that the FPU equation with non-linear cubic terms in one dimension shows $f^{-\mu}$ fluctuation in the appropriate non-linearity and frequency range. We can see $f^{-\mu}$ fluctuations do not come from the large number of degrees of freedom since our FPU equation shows $f^{-\mu}$ PSD even in N = 8 small system. This enables us to investigate the general relation between $f^{-\mu}$ property and the motion in systems of small degrees of freedom.

In this paper we investigate FPU equation in the phase space of small degrees of freedom such as N=4, 6, .. where the non-linear term has comparable order with the linear term. We survey trajectories in the phase space motions both showing and not showing $f^{-\mu}$ PSD.

We use the following FPU equation:

$$\partial^2 u_i/\partial t^2 = u_{i+1} - 2u_i + u_{i-1} + \lambda\{(u_{i+1} - u_i)^Q - (u_i - u_{i-1})^Q\} \tag{1}$$

where u_i is the displacement of ith site, Q is 3 in our case and i runs from 1 to N. We adopt the periodic boundary condition. The FPU equation starts from the initial condition that the initial u_i is set as a random number between -1 and 1 and the initial velocity is set to zero at all sites. It looks that this specific initial condition does not seem to affect the generality of conclusions in our experiments.

As a computational method, we use Symplectic Integrator (SI) [11]. SI is known as an effective computational method for symplectic forms [12] which Hamiltonian systems belong

to. Phenomenological estimate of computational errors of the amplitude is an order of about 10^{-9} and the energy is conserved at an accuracy of 10^{-12} for our condition e.g. time step $= 0.01$.

Following the previous letter [1], we use the modal energy or the Fourier component $a_n = \sum_j u_j exp(2\pi\sqrt{-1}nj/N)$ of the amplitude u_j as a physical quantity which corresponds to the phonon number in the classical limit. The $|a_n|^2$ oscillates regularly but sometimes bursts. $|a_n|^2$ shows $f^{-\mu}(\mu \sim 1)$ fluctuations in the small frequency f region of PSD.

We can characterize the behavior by dividing the time sequence into two states and make a symbolic sequence composed of two states e.g. 0100000010000... where '1' is 'burst' where the fluctuation exceeds a threshold and otherwise the state is labeled as '0'. Define the pausing time 'T' as the period between neighboring bursts. The distribution function $P(T)$ of T is $P(T) \sim T^{-2}$ when the PSD is f^{-1}. It is known in chaos theory of small degrees of freedom that $P(T)$ is $P(T) \sim T^{-2}$ when f^{-1} PSD is observed [13]. Our result is well consistent with chaos theory of small degrees of freedom.

There are tori and chaotic regions in conservative systems [13]. Our knowledge based upon maps of small degrees of freedom suggests $f^{-\mu}$ properties appear when the trajectory migrates among tori and chaotic regions in appropriate ways. It is of benefit to know the distribution of tori and chaotic regions in the phase space to interpret the motion.

FIGURES

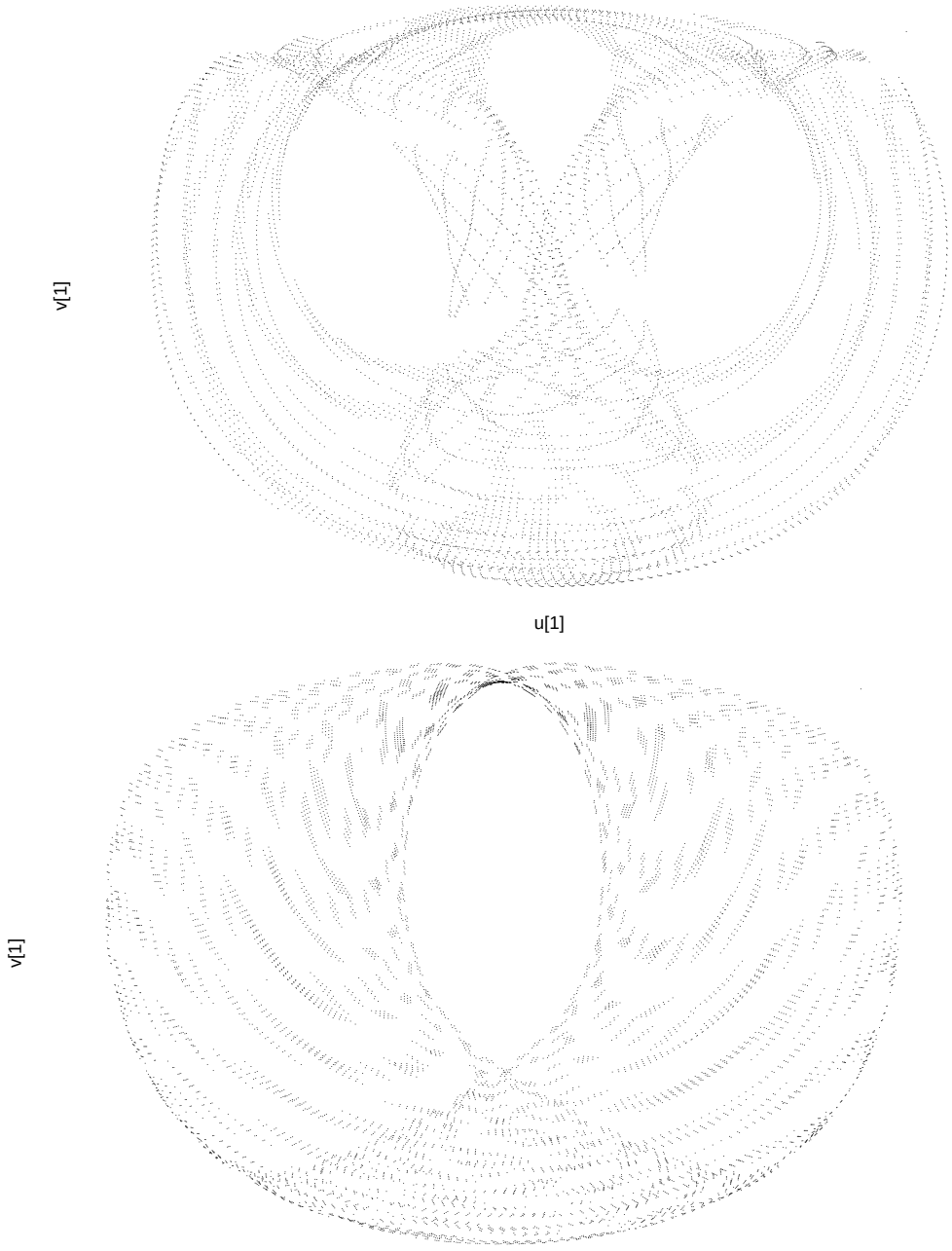

FIG. 1. The upper figure(a) is f^{-1} case, the lower one(b) is f^0. The points are where the trajectory crosses a 2 dimensional section defined by $u_0 = 0 \pm 0.01$. f^{-1} trajectory looks to be trapped in some complicated hierarchical structure, whereas f^0 one goes around over tori.

Firstly we investigate the phase space of system size $N = 4$. Almost all the trajectories show $f^{-\mu}(\mu \sim 0)$ PSD but rarely a trajectory shows a $f^{-\mu}(\mu \sim 1)$ PSD. See Fig. 1(a) where

the trajectory shows nearly f^{-1} PSD. The points in it are where the trajectory crosses a 2 dimensional section defined by $u_0 = 0 \pm 0.01$. Fig. 1(b) is obtained in the same way for a trajectory to show the white noise. We can see the trajectory in Fig. 1(a) is trapped in some complicated hierarchical structure but in Fig. 1 (b) the f^0 trajectory goes around all over tori, which looks pseudo-periodic in the white noise case. This result is instructive but not sufficiently general since we could not have found any other example than that shown in Fig. 1.

We investigate N=6 case as a more general case. We can see $f^{-\mu}$ PSD more than in N=4 case. The simple section taken as the same way as above is unclear. The large dimension disables us to show details of the phase space. Even N=6 makes it ambiguous. We introduce Lyapunov Spectrum which is powerful to describe the strength of instability at each point in the phase space.

The Lyapunov Spectrum is a characteristic of asymptotic orbital instability of dynamical systems. It is a set of real numbers with 2N elements $\lambda_1, ..., \lambda_{2N}$. The elements are mathematically defined by the eigenvalue spectrum of the squared Jacobi matrix by $J(t) = \frac{\partial(p(t),q(t))}{\partial(p(0),q(0))}$ and ${}^tJ(T)J(T) \sim \{\exp(2\lambda_1 T), ..., \exp(2\lambda_{2N} T)\}$. For an actual computation of the spectrum we use the development by the exterior derivative of Symplectic Integrator and standard Gram-Schmidt orthonormalization [14]. Rearranging the exponents of the spectrum we obtain $\lambda_1, ..., \lambda_{2N}$ in the decreasing order.

We calculate how fast the largest exponent λ_1 within some given period , which is short but longer than the characteristic period to go around the system. The temporal decay rate of Lyapunov exponents within the finite time t is $\sim 1/t$ when the initial points belong to tori [15]. We classify orbits depending upon a threshold of whether λ converges faster than $1/t^\alpha$ (e.g. set α as $0.3 \sim 0.5$). This criterion determines an initial condition belongs to tori($\leq 1/t^\alpha$) or chaotic regions($\sim$ const). Due to the restricted computational time, we could not apply this procedure to initial points chosen among all points on an energy-constant hypersurface but a $f^{-\mu}$ trajectory.

When $\lambda \sim 0$, we confirm that our procedure can draws tori like structure. When the

non-linear term is comparable with the linear term, a f^0 trajectory also looks to cover all the phase space without exception. It does not look trapped in some complicated structure. These results show the same ones in N=4 case, so confirm appropriate criterion to survey the phase space structure.

FIG. 2. Dots are initial conditions such that $\lambda_{max}(t) \leq 1/t^{0.3}$. The initial conditions are set along a $f^{-\mu}$ trajectory. It suggests hierarchical structure where tori surround tori like an onion.

See Fig. 2 where λ is 0.8, the threshold is $1/t^{0.3}$ and points on this figure belong to tori. The trace along a trajectory is projected to 2-dimensional plane. Tori surround tori like an onion [15]. One trajectory goes round and round but not visit regions near edges. There are few points in the upper and lower regions in Fig. 2. These are attached to regions where the modal energy burst and instability speculated by the finite time Lyapunov Exponent becomes large. These are considered to belong to chaotic regions. This coincidence suggests that $f^{-\mu}$ property directly reflects the motion in the phase space. We can see this result as that the trajectory goes around between tori and chaotic regions at some rate of migration.

Properties mentioned above correspond to results known in chaotic theory of maps for small degrees of freedom. It appears $f^{-\mu}$ trajectory migrates among hierarchical structures.

$f^{-\mu}$ properties require a certain complicated structure, so we could not see it in too small dimensional space. However it must require special conditions for e.g. ratio of structure hierarchy. For symplectic maps it is known that torus possession in the phase space decreases fast [16]. It is important to investigate this mechanism to generate $f^{-\mu}$ PSD is appropriate in $N \to \infty$. It is amusing to point out a possibility of another mechanism based on the relation between dissipation [3] and $f^{-\mu}$ properties but the evidences are yet insubstantial [8].

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